Solutions by Bracket

A First Course in Probability
Chapter 4—Problems

4. Five men and 5 women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman (for instance, $X = 1$ if the top-ranked person is female). Find $P\{X = i\}$, $i = 1, 2, 3, \ldots, 8, 9, 10$.

Let E_i be the event that the the ith scorer is female. Then the event $X = i$ correspdonds to the event $E_1^c E_2^c \cdots E_i$. It follows that

$$\begin{aligned} P\{X = i\} &= P(E_1^c E_2^c \cdots E_i) \\ &= P(E_1^c)P(E_2^c|E_1^c) \cdots P(E_i|E_1^c \cdots E_{i-1}^c) \end{aligned}.$$

Thus we have

i	$P\{X = i\}$
1	$^1/_2$
2	$^5/_{18}$
3	$^5/_{36}$
4	$^5/_{84}$
5	$^5/_{252}$
6	$^1/_{252}$
$7, 8, 9, 10$	0.

12. In the game of Two-Finger Morra, 2 players show 1 or 2 fingers and simultaneously guess the number of fingers their opponent will show. If only one of the players guesses correctly, he wins an amount (in dollars) equal to the sum of the fingers shown by him and his opponent. If both players guess correctly or if neither players guess correctly, then no money is exchanged. Consider a specified player and denote by X the amount of money he wins in a single game of Two-Finger Morra.

a. If each player acts independently of the other, and if each player makes his choice of the number of fingers he will hold up and the number he will guess that his opponent will hold up in such a way that each of the 4 possibilities is equally likely, what are the possible values of X and what are their associated probabilities?

A given player can only win 0, ± 2, ± 3, or ± 4 dollars. Consider two players A and B, and let X denote player A's winnings. Let A_{ij} denote the event that player A shows i fingers and guesses j, and define B_{ij} similarly for player B.

We have $P\{X = 2\} = P(A_{11}B_{12}) = P(A_{11})P(B_{12}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$, since we have assumed that A_{ij} and B_{ij} are independent and that $P(A_{ij}) = P(B_{ij}) = \frac{1}{4}$. Similarly, we have $P\{X = 3\} = P(A_{12}B_{22} \cup A_{21}B_{11}) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$ and $P\{X = 4\} = P(A_{22}B_{21}) = \frac{1}{16}$. Note that the situation is completely symmetric for player B, so the we have $P\{X = -2\} = P\{X = -4\} = \frac{1}{16}$ and $P\{X = -3\} = \frac{1}{8}$. Finally, we have $P\{X = 0\} = 1 - P\{X \neq 0\} = 1 - \frac{1}{2} = \frac{1}{2}$.

b. Suppose that each player acts independently of the other. If each player decides to hold up the same number of fingers that he guesses his opponent will hold up, and if each player is equally likely to hold up 1 or 2 fingers, what are the possible values of X and their associated probabilities?

Neither player can win any money in this scenario. If player A shows 1 finger and guesses B will show 1 finger, then A can only win if B shows 1 finger. But if B shows 1 finger, then B will guess that A will show 1 finger, and thus neither player will win. The same holds for when A shows 2 fingers and guesses that B will show 2 fingers. Thus, we have $P\{X = 0\} = 1$.

20. A gambling book recommends the following "winning strategy" for the game of roulette. It recommends that the gambler bet \$1 on red. If red appears (which has probability $\frac{18}{38}$), then the gambler should take her \$1 profit and quit. If the gambler loses this bet (which has probability $\frac{20}{38}$ of occurring), she should make additional \$1 bets on red on each of the next two spins of the roulette wheel and then quit. Let X denote the gambler's winnings when she quits.

a. Find $P\{X > 0\}$.

Note that X only takes on the values -3, -1, and 1. Thus

$$\begin{aligned} P\{X > 0\} &= P\{X = 1\} \\ &= P(\text{she wins immediately or she loses and then wins the next two}) \\ &= P(\text{she wins immediately}) + P(\text{she loses and then wins the next two}) \\ &= \frac{18}{38} + \frac{20}{38} \cdot \frac{18}{38} \cdot \frac{18}{38} \approx .592 \end{aligned}$$

b. Are you convinced that the winning strategy is indeed a "winning" strategy? Explain your answer!

The expected value of X is negative ($\approx -.108$), which is accounted for by the fact that although the gambler has a high probability of winning \$1, she could also lose \$3, and the probability of this occuring is not low enough to make the game worth playing in the long run.

21. A total of 4 buses carrying 148 students form the same school arrives at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

a. Which of $E[X]$ or $E[Y]$ do you think is bigger? Why?

We should expect $E[X]$ to be larger since it's the per-student average rather than the per-bus average, just as the per-student average class size was larger than the per-class average class size (from the example in class).

b. Compute $E[X]$ and $E[Y]$.

We have

$$\begin{aligned} E[X] &= \frac{25}{148} \cdot 25 + \frac{33}{148} \cdot 33 + \frac{40}{148} \cdot 40 + \frac{50}{148} \cdot 50 \approx 39.28 \\ E[Y] &= \frac{1}{4} \cdot 25 + \frac{1}{4} \cdot 33 + \frac{1}{4} \cdot 40 + \frac{1}{4} \cdot 50 = 37 \end{aligned}$$

27. An insurance company writes a policy to the effect that an amount of money A must be paid if some event E occurs within a year. If the company estimates that E will occur within a year with probability p, what should it charge the customer in order that its expected profit will be 10 percent of A?

Let X be denote the company's profit at the end of the year, and w be the amount that the customer is charged. The company's profit is w if E does not occur within the year, and $w - A$ if E does occur within the year. Thus $P\{X = w\} = (1 - p)$ and $P\{X = w - A\} = p$. Therefore $E[X] = w(1 - p) + (w - A)p = w - Ap$. We set $E[X] = .1A$ to obtain $w = A(p + .1)$.

31. Each night different meteorologists give us us the probability that it will rain the next day. To judge how well these people predict, we will score each of them as follows: If a meteorologist says that it will rain with probability p, then he or she will receive a score of

$$\begin{aligned} 1-(1-p)^2 &\qquad \text{if it does rain,} \\ 1-p^2 &\qquad \text{if it does not rain.} \end{aligned}$$

We will then keep track of scores over a certain time span and conclude that the meteorologist with the highest average score is the best predictor of weather. Suppose now that a given meteorologist is aware of this and wants to maximize his or her expected score. If this person truly believes that it will rain tomorrow with probability p^*, what value of p should he or she assert so as to maximize the expected score?

Let X be the score that the meteorologist receives, given that she has asserted that it will rain tomorrow with probability p. Then $P\{X = [1-(1-p)^2]\} = p^*$ and $P\{X = (1-p^2)\} = (1-p^*)$. It follows that $E[X] = [1-(1-p)^2]p^* + (1-p^2)(1-p^*)$, which we rearrange and write as a function of p to obtain $E[X] = f(p) = -p^2 + 2p^*p + 1 - p^*$. We differentiate with respect to p to obtain $f'(p) = -2p + 2p^*$, which clearly has a zero at $p = p^*$. It is straightforward to verify that f has a maximum at this zero, so the meteorologist should assert $p = p^*$ as the probability that it will rain tomorrow.

41. A man claims to have extrasensory perception. As a test, a fair coin is flipped 10 times, and the man is asked to predict the outcome in advance. He gets 7 out of 10 correct. What is the probability that he would have done at least this well if he had no ESP?

If the man were just guessing, then on each flip he would have probability $p = \frac{1}{2}$ of getting the correct answer. Let X be the number of correct guesses out of a sequence of 10 coin flips, and we can see that X is a binomial random variable with parameters 10 and $\frac{1}{2}$. Thus $P\{X \geq 7\} = \sum_{i=7}^{10} \binom{10}{i} (\frac{1}{2})^i (\frac{1}{2})^{10-i} = \frac{11}{64}$.

51. The expected number of typographical errors on a page of a certain magazine is .2. What is the probability that the next page you read contains (a)0 and (b)2 or more typographical errors? Explain your reasoning.

Let X be the number of typographical errors on a page of a magazine. Then X is a Poisson random variable with parameter $\lambda = E[X] = .2$. We then have $P\{X = 0\} = e^{-.2} \approx .819$ and $P\{X \geq 2\} = 1 - P\{X < 2\} = 1 - P\{X = 0\} - P\{X = 1\} = 1 - e^{-.2} - .2e^{-.2} \approx .0175$.

57. Suppose that the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$.

a. Find the probability that 3 or more accidents occur today.

Let X denote the number of accidents on the stretch of road. Then $P\{X \geq 3\} = 1 - P\{X < 3\} = 1 - e^{-3} - 3e^{-3} - \frac{9}{2}e^{-3} \approx .577$.

b. Repeat part (a) under the assumption that at least 1 accident occurs today.

Note that that the event "there are three or more accidents today," is a subset of the event "there is at least one accident today," and thus the intersection of the two is just the former. It follows that

$$P\{X \geq 3 | X \geq 1\} = \frac{P\{X \geq 3\}}{P\{X \geq 1\}} = \frac{1 - e^{-3} - 3e^{-3} - \frac{9}{2}e^{-3}}{1 - e^{-3}} \approx .607.$$

63. People enter a gambling casino at a rate of 1 for every 2 minutes.

a. What is the probability that no one enters between 12:00 and 12:05?

If X is the number of people entering within the 5 minute interval, then X is a Poisson random variable with parameter $\lambda = \frac{1}{2} \cdot 5$. Thus, $P\{X = 0\} = e^{-\frac{5}{2}} \approx .082$.

b. What is the probability that at least 4 people enter the casino during that time?

Using the same random variable as above, we have

$$P\{X \geq 4\} = 1 - e^{-\frac{5}{2}} - \frac{5}{2}e^{-\frac{5}{2}} - \frac{25}{4 \cdot 2!}e^{-\frac{5}{2}} - \frac{125}{8 \cdot 3!}e^{-\frac{5}{2}} \approx .242$$

68. In response to an attack of ten missiles, five hundred antiballistic missiles are launched. The missile targets of the antiballistic missiles are independent, with each being equally likely to go towards any of the missiles. If each antiballistic missile independently hits its target with probability .1, use the Poisson paradigm to approximate the probability that all missiles are hit.

Consider one particular missile M. A particular antiballistic missile A selects M as its target with probability .1, and if A selects M then it has probability .1 of hitting it. Hence any such A will hit M with probability $(.1)(.1) = .01$. Then the likely number of times M gets hit is roughly $500(.01) = 5$. Hence by the Poisson paradigm, if X is M's likely number of hits then X is a Poisson(5) variable. Thus the probability that M is hit is $P\{X > 0\} = 1 - P\{X = 0\} = 1 - e^{-5}$. There are 10 missiles, so the probability that all of them are hit is then roughly $(1 - e^{-5})^{10}$.

71. Consider a roulette wheel consisting of 38 numbers—1 through 36, 0, and double 0. If Smith always bets that the outcome will be one of the numbers 1 through 12, what is the probability that

a. Smith will lose his first 5 bets;

Since Smith will lose with probability $\frac{26}{38}$, we will lose his first 5 bets with probability $(\frac{13}{19})^5 \approx .15$.

b. his first win will occur on his 4th bet?

Note that this is a geometric random variable with parameter $p = \frac{12}{38}$ (or alternatively, a negative binomial random variable with parameters $p = \frac{12}{38}$ and $r = 1$). Smith's first win will occur on his 4th bet with probabiltity $(\frac{13}{19})^3 \cdot \frac{6}{19} \approx .101$.

75. A fair coin is continually flipped until heads appears for the tenth time. Let X denote the number of tails that occur. Compute the probability mass function of X.

Let Y be a negative binomial random variable with parameters $p = \frac{1}{2}$ and $r = 10$. An appropriate sequence with n tails in it must contain $n + 10$ flips in it total, and thus

$$P\{X = n\} = P\{Y = n + 10\} = \binom{(n + 10) - 1}{r - 1} p^r (1 - p)^{(n+10)-r} = \binom{n + 9}{9} \left(\frac{1}{2}\right)^{n+10}$$